A note on orthogonality of subspaces in Euclidean geometry

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Abstract

We show that Euclidean geometry in suitably high dimension can be expressed as a theory of orthogonality of subspaces with fixed dimensions and fixed dimension of their meet.

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1 Introduction

While the notion of orthogonality of lines in Euclidean geometry has well founded meaning (it is frequently used as a primitive notion, see [2]), orthogonality of subspaces can be defined in several different ways. Two of them were shown in [5] to be sufficient in Euclidean geometry; actually, each of these two considered on the universe of subspaces of fixed dimension can be used to reinterpret the underlying point-line affine space and after that to define line orthogonality. Thus the procedure of reinterpretation consists, in fact, in two steps and in the second step one should define orthogonality of lines in terms of a given orthogonality of subspaces. In this note we show that such a definition is possible for each prescribed values of dimensions of the considered subspaces (Theorem 2.4(ii)).

The notion of orthogonality of subspaces is not a unique-meaning relation, even if dimensions of the subspaces involved are fixed. Therefore, we have to deal with a family of possible relations of orthogonality. And in this note we show that *each one* of these relations is sufficient to express the underlying geometry provided the latter has sufficiently high dimension (Theorem 2.4(i)).

So, finally, we prove that Euclidean geometry can be expressed in the language with points, subspaces (of fixed dimensions), and orthogonality of subspaces. It is a folklore that affine geometry can be expressed as a theory of point-k-subspace incidence. Euclidean geometry appears when we impose a relation of orthogonality on that "affine" structure.

Our result does not solve the problem whether Euclidean geometry can be expressed in the language with k-subspaces *as individuals* and some of the orthogonalities introduced above as a single primitive notion, in that way, possibly, generalizing [5]. We conjecture that the answer is affirmative, but the question is addressed in other papers.

We close the paper with a list of some more interesting properties of the orthogonalities considered here. This list is not intended as a complete axiom system, but we think that at least some of its items can be used to build such a system characterizing orthogonality of subspaces.

2 Results

Let $\mathfrak{M} = \langle S, \mathcal{L}, \perp\rangle$ be an Euclidean space, where $\mathfrak{A} := \langle S, \mathcal{L}\rangle$ is an affine space with $\mathcal{L} \subset 2^S$ and $\perp \subset \mathcal{L} \times \mathcal{L}$ is a line orthogonality (cf. [2]). Up to an isomorphism $\mathfrak{M}$ corresponds to $\langle V, \mathcal{L}_V, \perp_\xi\rangle$ where V is a vector space, $\mathcal{L}_V$ is the set of translates of 1-dimensional subspaces of V and $\perp_\xi$ is the orthogonality determined by a nondegenerate symmetric bilinear form ξ on V with no isotropic directions. For each nonnegative integer k, $\mathcal{H}_k$ stands for the class of all k-dimensional subspaces of $\mathfrak{M}$, and $\mathcal{H}$ stands for all subspaces of $\mathfrak{M}$. If $X_1, X_2 \in \mathcal{H}$ we write $X_1 \sqcup X_2$ for the least subspace in $\mathcal{H}$ that contains $X_1 \cup X_2$ (i.e. the meet of all elements of $\mathcal{H}$ containing $X_1 \cup X_2$). Note an evident fact that follows from elementary affine geometry.

FACT 2.1. (i) *The family $\mathcal{H}_k$ is definable in $\mathfrak{A}$ for each nonnegative integer k.*

(ii) *Let $k < \dim(\mathfrak{A})$. Then the family $\mathcal{L}$ is definable in the incidence structure $\langle S, \mathcal{H}_k\rangle$. Consequently, $\mathfrak{A}$ is definable in $\langle S, \mathcal{H}_k\rangle$.*

Recall that $\mathfrak{M}$ is definitionally equivalent to the structure $\langle S, \mathcal{L}, \perp\!\!\!\perp\rangle$ (cf. e.g. an axiom system for $\perp\!\!\!\perp$ in [4], [7]), where $\perp\!\!\!\perp \subset S^2 \times S^2$ is defined in $\mathfrak{M}$ by the formula

$$a, b \perp\!\!\!\perp c, d :\iff \text{ there are } L_1, L_2 \in \mathcal{L} \text{ such that } a, b \in L_1 \perp L_2 \ni c, d. \tag{1}$$

Given any two $X, Y \in \mathcal{H}$ we write

$$X \perp Y :\iff a, b \perp\!\!\!\perp c, d \text{ for all } a, b \in X,\ c, d \in Y. \tag{2}$$

Note that for $X, Y \in \mathcal{L}$ the relation defined by (2) coincides with the orthogonality we have started from. If $X \perp Y$ then $X \cap Y$ is at most a point; we write

$$X \perp^* Y :\iff X \perp Y \text{ and } X \cap Y \neq \emptyset. \tag{3}$$

Recall that for any two subspaces $X, V \in \mathcal{H}$ such that $X \subset V$ and a point $q \in X$ there is the unique maximal $X' \in \mathcal{H}$ such that $q \in X' \perp^* X$, $X' \subset V$, and $X \sqcup X' = V$. We call X' an *orthocomplement* of X in V through q. If X is a point then necessarily $X = \{q\}$ and $X' = V$.

Let us define now (cf. Figure 2.1)

$$\begin{aligned} X_1 \mathrel{\underline{\phi}} X_2 \ :\iff\ & \text{there is a point } q \in X_1 \cap X_2 \text{ and } Z_1, Z_2 \in \mathcal{H} \text{ such that} \\ & q \in Z_1, Z_2 \perp^* X_1 \cap X_2,\ \ Z_1 \perp^* Z_2 \text{ and } (X_1 \cap X_2) \sqcup Z_i = X_i \text{ for } i = 1, 2. \end{aligned} \tag{4}$$

It is seen that the relation $\mathrel{\underline{\phi}}$ is symmetric. It is also not too hard to note that the following holds

$$X_1 \mathrel{\underline{\phi}} X_2 \iff \text{there is } Z_i \in \mathcal{H} \text{ such that } Z_i \perp^* X_{3-i} \text{ and } (X_1 \cap X_2) \sqcup Z_i = X_i \tag{5}$$

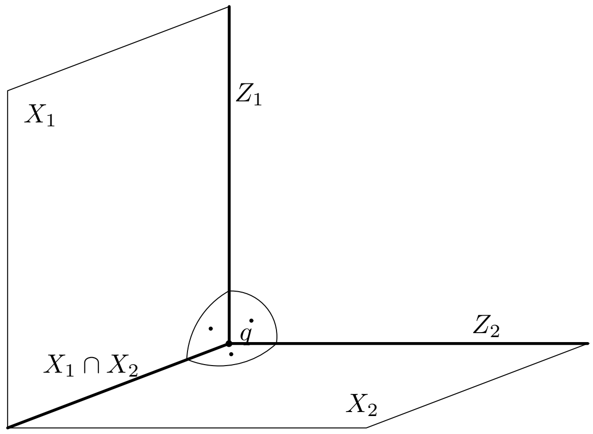

Figure 2.1

for both $i = 1, 2$. Note that when $X_1 \cap X_2$ is a point then $X_1 \underline{\perp} X_2$ and $X_1 \perp^* X_2$ are equivalent. Recall also a known formula

$$q \in X_1, X_2 \perp^* Y \ni q \implies Y \perp^* (X_1 \sqcup X_2). \tag{6}$$

The motivation for such general definition (4) is reflection geometry (cf. [1], [6]). Denote by σ_X the reflection in a subspace X, i.e. an involutory isometry that fixes X pointwise; then

$$\sigma_{X_1}\sigma_{X_2} = \sigma_{X_2}\sigma_{X_1} \iff X_1 \perp\!\!\!\!\circ\, X_2. \tag{7}$$

One might call $\perp\!\!\!\!\circ$ an orthogonality, but note that (7) yields the formula

$$X_1 \subset X_2 \implies X_1 \perp\!\!\!\!\circ\, X_2, \tag{8}$$

which fails to fit intuitions that are commonly associated with the notion of an orthogonality of subspaces in an Euclidean space. For this reason we put some restrictions on $\perp\!\!\!\!\circ$ to get a relation that conforms intuitions concerning Euclidean orthogonality more:

$$X_1 \underline{\perp} X_2 \;\; :\iff X_1 \perp\!\!\!\!\circ\, X_2 \text{ and } X_1 \cap X_2 \neq X_1, X_2. \tag{9}$$

So, in view of (8) the relations $\underline{\perp}$ and $\perp\!\!\!\!\circ$ are closely related indeed:

$$X_1 \perp\!\!\!\!\circ\, X_2 \iff X_1 \underline{\perp} X_2 \text{ or } X_1 \subset X_2 \text{ or } X_2 \subset X_1. \tag{10}$$

In view of (4), (5) it is seen that the relation $X_1 \underline{\perp} X_2$ can be characterized by any of the following three (mutually equivalent) conditions:

(4.0) there is a point $q \in X_1 \cap X_2$ and $Z_1, Z_2 \in \mathcal{H} \setminus \mathcal{H}_0$ such that
$q \in Z_1, Z_2 \perp^* X_1 \cap X_2$, $Z_1 \perp^* Z_2$, and $(X_1 \cap X_2) \sqcup Z_i = X_i$ for $i = 1, 2$;

(5.i) there is $Z_i \in \mathcal{H} \setminus \mathcal{H}_0$ such that
$Z_i \perp^* X_{3-i}$, $(X_1 \cap X_2) \sqcup Z_i = X_i$, and not $X_{3-i} \subset X_i$;

where $i = 1, 2$.

Let us write

$$X_1 \perp^m_{k_1,k_2} X_2 \text{ when } X_1 \mathrel{\underline{\perp}} X_2,\ X_1 \in \mathcal{H}_{k_1},\ X_2 \in \mathcal{H}_{k_2}, \text{ and } X_1 \cap X_2 \in \mathcal{H}_m.$$

Following this terminology we can say that the orthoadjacency relation $\mathrel{\underset{\sim}{\perp}}_k$ considered in [5] is the relation $\perp^{k-1}_{k,k}$ for a fixed integer k.

Note the evident restrictions that dimensions k_1, k_2, m must satisfy in order to have $\perp^m_{k_1,k_2}$ nontrivial

$$\text{there are } X_1, X_2 \text{ such that } X_1 \perp^m_{k_1,k_2} X_2 \quad \Longleftrightarrow \quad k_1 + k_2 - m \le \dim(\mathfrak{M}). \tag{11}$$

LEMMA 2.2. *Let $Y_1 \in \mathcal{H}_{k_1-m}$ and $X_2 \in \mathcal{H}_{k_2}$ intersect in a point. Assume that $k_1 \le k_2$ and $k_1 + k_2 - m \le \dim(\mathfrak{M})$. The following conditions are equivalent*

(i) $Y_1 \mathrel{\underline{\perp}} X_2$ *(i.e. actually, $Y_1 \perp^* X_2$);*

(ii) $X_1 \mathrel{\underline{\perp}} X_2$ *for each $X_1 \in \mathcal{H}_{k_1}$ such that $Y_1 \subset X_1$ and* $\dim(X_1 \cap X_2) = m$.

PROOF. The implication (i) $\Longrightarrow$ (ii) follows directly from (5.1).

Assume (ii); set $V := Y_1 \sqcup X_2$ and let $q \in Y_1 \cap X_2$. Then $\dim(V) = k_1 + k_2 - m$. Let W be the orthocomplement of Y_1 in V through q, so $\dim(W) = k_2$. Since $k_1 \le k_2$ we have $m \le \dim(W \cap X_2)$, so there is $T \subset W \cap X_2$ with $\dim(T) = m$. Set $X_1 := T \sqcup Y_1$. Then $\dim(X_1) = k_1$ and thus $X_1 \mathrel{\underline{\perp}} X_2$. Clearly, $X_1 \cap X_2 = T$. By (5.1), there is $Z \in \mathcal{H}$ such that $Z \perp^* X_2$ and $X_1 = T \sqcup Z$. Since both Y_1, Z are orthocomplements of T in X_1, we get $Z = Y_1$ and thus (i) follows. $\square$

LEMMA 2.3. *Let $1 \le k_1$ and $1 < k_2$. Then for $L_1, L_2 \in \mathcal{L}$ the following conditions are equivalent*

(i) $L_1 \perp L_2$*;*

(ii) *there are $X_1 \in \mathcal{H}_{k_1}$, $X_2 \in \mathcal{H}_{k_2}$ such that $X_1 \perp^* X_2$ and $L_i \subset X_i$ for $i = 1, 2$.*

Notice that the assumption $1 < k_2$ in 2.3 is significant as the lines L_1, L_2 could be skew so, we need some more room in X_2 to find there the translate of L_2 that meets L_1.

Now, let us consider the structure

$$\mathfrak{K} := \langle S, \mathcal{H}_{k_1}, \mathcal{H}_{k_2}, \mathrel{\underline{\phi}} \cap (\mathcal{H}_{k_1} \times \mathcal{H}_{k_2})\rangle;$$

for fixed k_1, k_2 such that $1 \le k_1, k_2 < \dim(\mathfrak{M})$. As the inclusion relations involved in (10) and (9) are expressible in terms of pure incidence language of $\langle S, \mathcal{H}_{k_1}, \mathcal{H}_{k_2}\rangle$ it is easily seen that $\mathfrak{K}$ and $\langle S, \mathcal{H}_{k_1}, \mathcal{H}_{k_2}, \mathrel{\underline{\perp}} \cap (\mathcal{H}_{k_1} \times \mathcal{H}_{k_2})\rangle$ are definitionally equivalent.

THEOREM 2.4. *Let $1 \le k_1, k_2$.*

(i) *If $k_1 + k_2 - m \le \dim(\mathfrak{M})$, then the Euclidean space $\mathfrak{M}$ is definable in the structure $\langle S, \mathcal{H}_{k_1}, \mathcal{H}_{k_2}, \perp^m_{k_1,k_2}\rangle$.*

(ii) *The Euclidean space $\mathfrak{M}$ is definable in $\mathfrak{K}$.*

PROOF. By 2.1, for each integer n the set $\mathcal{H}_n$ is definable in the reduct $\langle S, \mathcal{H}_{k_1}, \mathcal{H}_{k_2}\rangle$ of $\mathfrak{K}$. In particular, the family $\mathcal{L}$ of lines of $\mathfrak{M}$ is definable in $\mathfrak{K}$. Moreover, $\perp^m_{k_1,k_2}$ is definable in $\mathfrak{K}$ for each sensible m. Without loss of generality we can assume that $k_1 \le k_2$. By 2.2, the relation $\perp^0_{k_1-m,k_2}$ is definable in $\mathfrak{K}$ and in $\langle S, \mathcal{H}_{k_1}, \mathcal{H}_{k_2}, \perp^m_{k_1,k_2}\rangle$. Finally, by 2.3 the proof is complete. $\square$

3 Synthetic properties of orthogonalities

In this section we aim to show a few specific properties of orthogonality relations $\underline{\Phi}$ and $\underline{\perp}$ considered on the family of all the subspaces of $\mathfrak{M}$. Some of them are analogous to known properties of the relation $\perp$ considered on the lines of $\mathfrak{M}$, but there are also remarkable differences.

3.1 Orthogonality $\underline{\perp}$

Fact 3.1. *Let $A, B, C \in \mathcal{H}$.*

(i) *If $A \underline{\perp} B$, then $B \underline{\perp} A$.*

(ii) *If $A \underline{\perp} B$, then $A \cap B \neq \emptyset$.*

(iii) *If $A \underline{\perp} B \parallel C$ and $A \cap C \neq \emptyset$, then $A \underline{\perp} C$.*

(iv) *There are no nonempty $D_1, D_2 \in \mathcal{H} \setminus \mathcal{H}_0$ with $D_1 \subseteq D_2$, and $D_1 \underline{\perp} D_2$.*

(v) *If $\emptyset \neq A \subsetneq B \subsetneq C$, then there is the unique $B' \in \mathcal{H}$ such that $B \cap B' = A$, $B \underline{\perp} B'$, and $B \sqcup B' = C$.*

Proposition 3.2. *Let $A, B, C \in \mathcal{H}$. If $A \underline{\perp} B$ and $A \underline{\perp} C$, then $A \underline{\perp} (B \sqcup C)$ or $A \subseteq B \sqcup C$.*

Proof. Assume that $A \underline{\perp} B$, $A \underline{\perp} C$, and $A \nsubseteq B \sqcup C$. By 3.1(i) we have $A \cap B \neq \emptyset$. Since $A \cap B \subseteq A \cap (B \sqcup C)$ there is a common point q of A and $B \sqcup C$. From our assumption and (5.1) there are $Z_B, Z_C \in \mathcal{H}$ such that

$$Z_B \perp^* A, \quad (A \cap B) \sqcup Z_B = B \qquad \text{and} \qquad Z_C \perp^* A, \quad (A \cap C) \sqcup Z_C = C. \tag{12}$$

Take $Z := Z'_B \sqcup Z'_C$, where Z'_B, Z'_C are translates of Z_B, Z_C respectively, through q. Therefore, by (6) and (12) we have $A \perp^* Z'_B \sqcup Z'_C = Z$. Now as $q \in A, B \sqcup C, Z$ and $Z \subseteq B \sqcup C$ we have $Z \sqcup (A \cap (B \sqcup C)) = (Z \sqcup A) \cap (B \sqcup C)$. Note that the equalities in (12) give $Z'_B \sqcup A = Z_B \sqcup A = B \sqcup A$ and $Z'_C \sqcup A = Z_C \sqcup A = C \sqcup A$. So, we have $Z \sqcup A = (B \sqcup A) \sqcup (C \sqcup A) = (B \sqcup C) \sqcup A$ and finally $Z \sqcup (A \cap (B \sqcup C)) = B \sqcup C$ which by (5.2) gives our claim. $\square$

In some specific cases $\underline{\perp}$ may be transitive under inclusion which is showed in next two propositions.

Proposition 3.3. *Let $A, B, C \in \mathcal{H}$. If $A \underline{\perp} B$ and $A \cap B \subsetneq C \subset B$, then $A \underline{\perp} C$.*

Proof. Let $q \in A \cap B$. From assumptions, $A \cap C = A \cap B$. By (5.1), there is $A' \in \mathcal{H}$ with $q \in A'$, $A = (A \cap B) \sqcup A'$ and $A' \perp^* B$. Thus $A' \perp^* C$ and the claim follows by (5.1). $\square$

Proposition 3.4. *Let $A, B, C \in \mathcal{H}$. If $A \underline{\perp} B$ and $A \cap B \subset C \subsetneq A$, then $A \underline{\perp} (B \sqcup C)$.*

PROOF. Let $q \in A \cap B$. Note that A, B, C lay in the bundle through q, i.e. in a projective space, and thus we have $C = (A \cap B) \sqcup C = A \cap (B \sqcup C)$. In view of (5.1) there is $Z \in \mathcal{H}$ such that $Z \perp^* A$ and $B = (A \cap B) \sqcup Z$. From the latter equality we have

$$B \sqcup C = (A \cap B) \sqcup Z \sqcup C = A \cap (B \sqcup C) \sqcup Z$$

which, together with $Z \perp^* A$, again by (5.1) completes the proof. $\square$

The following example shows that it is hard to tell anything more about transitivity of $\underline{\perp}$ than it is said in 3.3 and 3.4.

EXAMPLE 3.5.

(i) There are $A, B, C \in \mathcal{H}$ such that

$$A \,\underline{\perp}\, B \subset C, \quad \neg A \,\underline{\perp}\, C, \quad \text{and} \quad \dim(C) = \dim(B) + 1.$$

(ii) There are $A, B, C \in \mathcal{H}$ such that

$$A \,\underline{\perp}\, B \supset C, \quad \neg A \,\underline{\perp}\, C, \quad A \cap C \neq \emptyset, \quad \text{and} \quad \dim(C) = \dim(B) - 1.$$

In essence, one can take lines A, B and a plane C in (i), as well as, planes A, B and a line C in (ii).

Therefore no "simple" form of transitivity can be proved. We finish with yet another property of $\underline{\perp}$.

PROPOSITION 3.6. *Let $A, B, C \in \mathcal{H}$. If $A \,\underline{\perp}\, B$, $A \,\underline{\perp}\, C$, and $A \cap B \cap C \neq \emptyset$, then $A \,\underline{\perp}\, (B \cap C)$ or $B \cap C \subseteq A$.*

PROOF. We can assume that $B \cap C$ is at least a line as otherwise our claim is clear. Let $q \in A \cap B \cap C$. Thanks to (5.1) we can take $Z_B, Z_C \in \mathcal{H}$ such that

$$Z_B \perp^* B, \quad (A \cap B) \sqcup Z_B = A \qquad \text{and} \qquad Z_C \perp^* C, \quad (A \cap C) \sqcup Z_C = A. \tag{13}$$

Note that Z_B is the orthocomplement of $A \cap B$ in A through q and Z_C is the orthocomplement of $A \cap C$ in A through q. So, slightly abusing notation we can write

$$Z := Z_B \sqcup Z_C = (A \cap B)^{\perp} \sqcup (A \cap C)^{\perp} = (A \cap B \cap C)^{\perp}.$$

Hence $Z \sqcup (A \cap B \cap C) = A$. Moreover $q \in Z_B, Z_C \perp^* B \cap C \ni q$ by (13). Hence by (6) we get $Z \perp^* B \cap C$, which in view of (5.1) suffices as a final argument. $\square$

3.2 Orthogonality $\underline{\phi}$

According to (9) or (10), properties of relation $\underline{\phi}$ are simple consequences of properties of relation $\underline{\perp}$ with possible inclusions between its arguments taken into account.

PROPOSITION 3.7. *Let $A, B, C \in \mathcal{H}$.*

(i) *If $A \,\underline{\phi}\, B$, then $B \,\underline{\phi}\, A$.*

(ii) *If $A \,\underline{\phi}\, B$, then $A \cap B \neq \emptyset$.*

(iii) *If* $A \mathrel{\underline{\Phi}} B \parallel C$ *and* $A \cap C \neq \emptyset$, *then* $A \mathrel{\underline{\Phi}} C$.

(iv) *If* $\emptyset \neq A \subsetneq B \subsetneq C$, *then there is the unique* $B' \in \mathcal{H}$ *such that* $B \cap B' = A$, $B \mathrel{\underline{\Phi}} B'$, *and* $B \sqcup B' = C$.

(v) *If* $A \mathrel{\underline{\Phi}} B$ *and* $A \mathrel{\underline{\Phi}} C$, *then* $A \mathrel{\underline{\Phi}} (B \sqcup C)$.

(vi) *If* $A \mathrel{\underline{\Phi}} B$ *and* $A \cap B \subset C \subset B$, *then* $A \mathrel{\underline{\Phi}} C$.

(vii) *If* $A \mathrel{\underline{\Phi}} B$ *and* $A \cap B \subset C \subset A$, *then* $A \mathrel{\underline{\Phi}} (B \sqcup C)$.

(viii) *If* $A \mathrel{\underline{\Phi}} B$, $A \mathrel{\underline{\Phi}} C$, *and* $A \cap B \cap C \neq \emptyset$, *then* $A \mathrel{\underline{\Phi}} (B \cap C)$.

PROOF. (i) – (iv) follow directly from 3.1 and (10).

(v): It suffices to apply (10) plus 3.2 or (8). Only two cases: (a) $C \subset A \mathrel{\underline{\perp}} B$, (b) $B \subset A \mathrel{\underline{\perp}} C$ of interpretation of the assumptions may appear problematic, but they are equivalent up to names of variables. Assume that (a) holds. Set $C' := C \sqcup (A \cap B)$. Then $C \sqcup B = C \sqcup (A \cap B) \sqcup B = C' \sqcup B$. So, we have $A \cap B \subset C' \subset A$. If $C' = A$, then the conclusion of (v) follows by (8). If $C' \neq A$ the claim follows by (3.4).

(vi) is immediate by (10) plus 3.3 or (8).

(vii) is immediate by (10) plus 3.4 or (8).

(viii): Apply (10) plus 3.6 or (8). Two cases, though equivalent up to variables, of the assumptions may raise some problems: (a) $B \mathrel{\underline{\perp}} A \subset C$ (b) $C \mathrel{\underline{\perp}} A \subset B$. Assume that (a) holds. Set $C' := B \cap C$. Then $A \cap B \subset C' \subset B$. If $A \cap B \neq C'$, then the claim comes from 3.3. If $A \cap B = C'$, then $B \cap C = C' = A \cap B \subset A$ and the claim is a consequence of (8). □

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